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(q, ν)-deformation of generalized basic hypergeometric states

M N Hounkonnou and E B Ngompe Nkouankam

International Chair in Mathematical Physics and Applications (ICMPA-UNESCO Chair) 072 BP: 50 Cotonou, Republic of Benin

E-mail: norbert_hounkonnou@cipma.net and hounkonnou@yahoo.fr

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Abstract

We provide with a (q, ν)-deformation of the generalized hypergeometric coherent states defined such that the normalization function is given by generalized basic hypergeometric functions. These states are eigenstates of suitably defined deformed lowering operators. We study the domain of convergence of the corresponding normalization function. On the basis of these states, we investigate generalized basic hypergeometric Husimi distributions and corresponding phase distributions as well as new analytic basis representations of arbitrary quantum states in Bargmann and Hardy spaces. The quantum statistical properties of the states, such as photon-counting statistics and quadrature squeezing are analytically and numerically discussed in the framework of conventional quantum optics.

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(Some figures in this article are in colour only in the electronic version)

1. introduction

Coherent states (CSs) of the quantum harmonic oscillator [1–3], as well as their various generalizations are so important in both mathematics and physics that they are worth studying in their own right. For salient features of the physical phenomena involved and their mathematical aspects, see [4–13].

Coherent states can be generalized using elegant and powerful algebraic methods and dynamical properties of the studied system as well as the supersymmetric quantum mechanics formalism. The system dynamics is usually exploited to construct CSs for systems with equally spaced energy levels. In such a method proposed by Neito and Simmons [14, 15], the position and momentum operators are built to give a harmonic oscillator like Hamiltonian. The coherent states are then determined as states minimizing a generalized uncertainty relation.

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In [16], generalized CSs are generated using an algebraic method, while in [17, 18] their construction is performed in the framework of the supersymmetric quantum mechanics (SUSYQM) formalism. These methods reveal to be efficient in the construction of coherent states for Morse [19, 20], the hydrogen atom [21], Eckart and Rosen–Morse [22] potentials as well as for hypergeometric-type functions [23]. Other generalizations also exist, elaborated as some special superpositions of canonical CSs. The known even and odd CSs [24], Shrödinger cat states [25, 26], etc can be so identified.

Furthermore, it becomes a matter of routine to distinguish the following classes of coherent state generalizations:

- The Barut–Giradello CSs [27] defined for the discrete series representations of the group $SU(1,1)$. These states can be realized in some physical systems such as the Pöschl–Teller and infinite square well potentials. Their generalization was performed by Gazeau and Klauder [11, 28–30], and Perelemov [31], separately. The key of the latter generalization lies on the idea that the construction of the oscillator CSs can be reformulated in the context of the group representation theory. Thus, Perelomov's CSs for a semi-simple Lie group G are points of an orbit of a unitary irreducible representation U of G in a Hilbert space $\mathcal{H}$.
- The Penson–Solomon generalized CSs [32] which are based on the generalization of the exponential function. These states include the Mittag-Leffler (ML) CSs [33], the Tricomi (TC) CSs [34] and the generalized hypergeometric (GH) CSs, shortly denoted by GHCSs in the sequel, introduced by Appl and Schiller [35]. These authors provide with a very large class of holomorphic CSs for which the normalization functions are given by generalized hypergeometric functions. These states have been extended to the mixed (thermal) states and applied to the case of pseudoharmonic oscillator [37].

Much to our very great surprise, it seems that all these remarkable coherent state generalizations, performed from the generalization of exponential function, can be generated (work in progress in our group), as particular cases, from a more general theory elaborated by Odzijewicz [38] in a nice, mathematically based work published in 1998, but unfortunately hushed up in the recent literature on the topic. In the mentioned work, this author investigated the quantum algebras generated by the coherent state maps of the disc, leading to a generalized analysis which includes standard analysis as well as q-analysis. He provided with the meromorphic continuation of the generalized basic hypergeometric series and constructed a reproducing measure, when the series is treated as a reproducing kernel.

The present study yields a (q,ν)-deformation of the GHCSs, obtained from the generalized bibasic hypergeometric series introduced in our previous work [39]:

$$
\begin{aligned}
{}_{\mu,s,m}^{\phi_1,\mathbf{d},\mathbf{b}}\Phi_{n,r,\nu}^{\mathbf{a},\mathbf{c},\phi_2}(z) = \sum_{l=0}^{+\infty} & \frac{(\mathbf{a};q)_l(\mathbf{c};p)_l}{\left(\frac{\phi_2(p,q)}{\phi_1(p,q)}q;q\right)_l(\mathbf{b};q)_l(\mathbf{d};p)_l} \\
& \times\left(\frac{p^{\mu+\nu}}{q^{\nu}}\right)^{l(l+1)/2}[(-1)^l q^{l(l-1)/2}]^{1+m-n}[(-1)^l p^{l(l-1)/2}]^{s-r}\left(\frac{z}{\phi_1(p,q)}\right)^l, \qquad (1)
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbf{a}=(a_1,\ldots,a_n) \qquad \mathbf{c}=(c_1,\ldots,c_r)\\
&\mathbf{b}=(b_1,\ldots,b_m) \qquad \mathbf{d}=(d_1,\ldots,d_s) \qquad (2)\\
&(\mathbf{a};q)_l=(a_1;q)_l\ldots(a_n;q)_l. \qquad (3)
\end{aligned}
$$

The use of conventions and notations on q-series [40] is adopted. The built states include a large class of well-known quantum states as particular or limiting cases.

The paper is organized as follows. In the following section, we first introduce the generalized basic hypergeometric states and study the domain of convergence and limit cases. In section 3, we deal with the resolution of unity and give a new analytic basis representations of arbitrary quantum states in Bargmann and Hardy spaces. In section 4, it is proved that the new GBHSs are eigenstates of suitably well-defined lowering operators. In section 5 we discuss the associated Husimi and Husimi phase distributions. Their geometrical and physical properties in quantum optics are studied in section 6. Finally, we end with concluding remarks in section 7.

2. Generalized basic hypergeometric states

In this section, we introduce the generalized hypergeometric states, classify them according to the criteria of convergence and specify the constraints on the parameters which the states depend on. Let us set $s = r, \mathbf{c} = \mathbf{d} = 0_{\mathbb{R}^r}$; $p = 1, \phi_1(1, q) = \phi_2(1, q) \equiv \phi(q)$ with $\lim_{q\to 1} \phi(q) = 1$ and $n = m + 1$. Then, (1) reduces to the generalized basic hypergeometric function

$$ {}^{\phi_1,0,\mathbf{b}}_{\mu,s,m}\Phi^{\mathbf{a},0,\phi_2}_{n,s,\nu}(z) \equiv {}^{\mathbf{b}}_{\nu}\Phi^{\mathbf{a}}_{\phi}(z) = \sum_{l=0}^{+\infty} \frac{(\mathbf{a}; q)_l}{(q; q)_l(\mathbf{b}; q)_l}(q^{-\nu})^{l(l+1)/2}\left(\frac{z}{\phi(q)}\right)^l, \tag{4}$$

and the generalized basic hypergeometric states (GBHSs) can be defined as

$$ |\mathbf{a}, \mathbf{b}; z\rangle^{\nu}_{\phi} = \left({}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}(|z|^2)\right)^{-1/2} \sum_{l=0}^{+\infty}(q^{-\nu})^{l(l+1)/4}\frac{z^l}{\sqrt{{}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(l)}}|l\rangle, \tag{5}$$

with the strictly positive parameter function

$$ {}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(l) = (q; q)_l\,\frac{(\mathbf{b}; q)_l}{(\mathbf{a}; q)_l}\phi^l. \tag{6}$$

One can readily check that the normalization function is given by

$$ {}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}(x) = {}^{\mathbf{b}}_{\nu}\Phi^{\mathbf{a}}_{\phi}(x) \qquad x = |z|^2. \tag{7}$$

The GBHSs depend on the complex variable z and on the set of numerator $(a_1, a_2, \ldots, a_{m+1})$ and denominator $(b_1, b_2, \ldots, b_m)$ parameters. For notation convenience, $(\mathbf{a}; q)_l = 1$ (resp. $(\mathbf{b}; q)_l = 1$) will be indicated by '$-$' instead of $\mathbf{a}$ (resp. $\mathbf{b}$).

In figure 1, as in the sequel, the graph (A) corresponds to the case $\mathbf{a} = -, \mathbf{b} = -$; $\phi(q) = 1$ and $\nu = -1$ while the graph (B) corresponds to the case $a_1 = 0, a_2 = q^2, a_i = 0$ for $i \geqslant 3$; $b_1 = q, b_i = 0$ for $i \geqslant 2$; $\phi(q) = 1$ and $\nu = -1$. This figure confirms that, under these conditions, the functions ${}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1 - q)x)$ reduce to $\exp(x)$ (the pointed line) as $q \to 1$.

Remark 2.1. Limit and particular cases.

(i) In the limit $q \to 1$, the states $|-, -; (1 - q)^{1/2}z\rangle^{\nu}_{\phi}$ reduce to the conventional states

$$ |z\rangle \equiv \mathrm{e}^{-|z|^2/2}\sum_{l=0}^{+\infty}\frac{z^l}{\sqrt{l!}}|l\rangle. $$

(ii) Taking $a_i = q^{\alpha_i}$ and $b_j = q^{\beta_j}$, $i = 1, 2, \ldots, m + 1$; $j = 1, 2, \ldots, m$, the states $|\mathbf{a}, \mathbf{b}; z\rangle$ reduce to $|m+1, m; z\rangle \equiv |\alpha_1, \alpha_2, \ldots, \alpha_{m+1}; \beta_1, \beta_2, \ldots, \beta_m; z\rangle_{(m+1,m)}$ when $q \to 1$, where

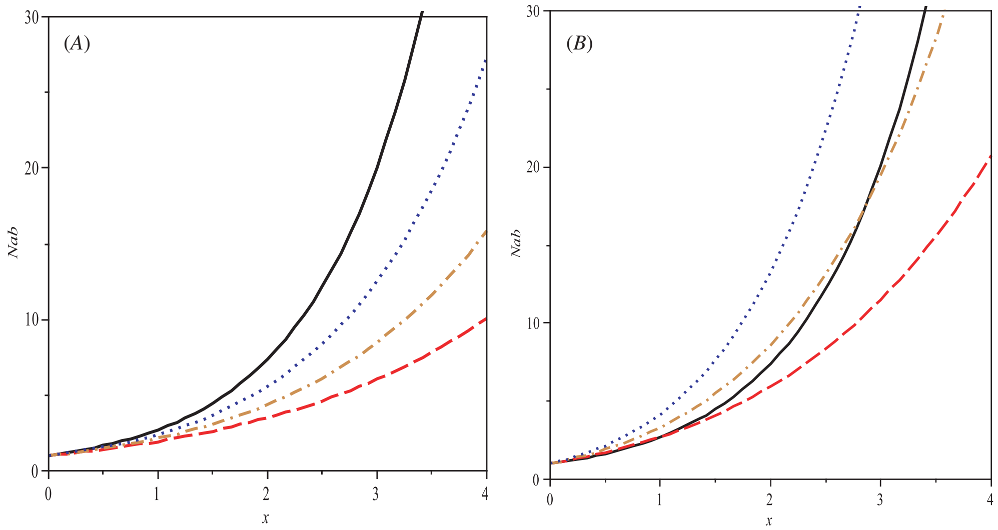

Figure 1. The normalization function ${}^{\mathbf{b}}_{-1}\mathcal{N}^{\mathbf{a}}_{1}((1-q)x) \equiv Nab(x)$ versus x for various q values: $q = 1$ (solid), $q = 0.7$ (dash), $q = 0.8$ (dashdot), $q = 0.9$ (dot).

$|p, r; z\rangle$ are the generalized hypergeometric states (GHS) defined by Appl and Schiller [35]:

$$\begin{aligned} |p, r; z\rangle &\equiv |a_1, \ldots, a_p; b_1, \ldots, b_r; z\rangle \\ &= [{}_p\mathcal{N}_r(|z|^2)]^{-1/2} \sum_{l=0}^{+\infty} \frac{z^l}{\sqrt{{}_p\rho_r(l)}} |l\rangle \end{aligned} \tag{8}$$

with

$${}_p\rho_r(l) = \Gamma(l+1) \frac{(b_1)_l \ldots (b_r)_l}{(a_1)_l \ldots (a_p)_l}, \tag{9}$$

where $(a)_l = \Gamma(a+l)/\Gamma(l)$ is the Pochhammer symbol, and the normalization function is given by the generalized hypergeometric function

$${}_p\mathcal{N}_r(\zeta) = \sum_{l=0}^{+\infty} \frac{(a_1)_l \ldots (a_p)_l}{(b_1)_l \ldots (b_r)_l} \frac{\zeta^l}{l!}. \tag{10}$$

(iii) The states $|-, -, (1-q)^{1/2} z\rangle_1^{-1}$ correspond to the coherent states introduced by Quesne [4]:

$$|z\rangle = [E_q((1-q)q|z|^2)]^{-1/2} \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{[n]_q^Q!}} |n\rangle \tag{11}$$

with

$$[n]_q^Q = \frac{1-q^{-n}}{q-1} \qquad E_q(z) = \prod_{k=0}^{+\infty} (1 + q^k z)$$
$$[n]_q^Q! \equiv [n]_q^Q [n-1]_q^Q \ldots [1]_q^Q.$$

(iv) The states $|-,-,(1-q)^{1/2}z\rangle_1^0$ correspond to the maths-type q-deformed coherent states [5]:

$$|z\rangle = \left[\exp_q^M(|z|^2)\right]^{-1/2} \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{[n]_q^M!}}|n\rangle \tag{12}$$

with

$$[n]_q^M = \frac{1-q^n}{1-q} \qquad \exp_q^M(x) = \sum_{n=0}^{+\infty} \frac{x^n}{[n]_q^M!}.$$

The states (5) thus appear as a generalization of various deformed states known in the literature. In particular, they yield a (q,ν)-deformation of the states given by Appl and Schiller in [35]. It is worth noting that although one-parameter deformations have been mostly studied, the multiparameter ones have aroused much interest because they become more flexible when we are dealing with applications to concrete physical models (see [36] and references therein).

The normalization function (7) defines the scalar product of two GBHSs with identical parameter sets:

$$\begin{aligned} {}_\phi^\nu\langle \mathbf{a}, \mathbf{b}; z'|\mathbf{a}, \mathbf{b}; z\rangle_\phi^\nu &= \left({}_\nu^{\mathbf{b}}\mathcal{N}_\phi^{\mathbf{a}}(|z|^2)\,{}_\nu^{\mathbf{b}}\mathcal{N}_\phi^{\mathbf{a}}(|z'|^2)\right)^{-1/2} \sum_{l=0}^{+\infty} \left(q^{-\nu}\right)^{l(l+1)/2} \frac{z'^{*l} z^l}{{}^{\mathbf{b}}\rho_\phi^{\mathbf{a}}(l)} \\ &= \frac{{}_\nu^{\mathbf{b}}\mathcal{N}_\phi^{\mathbf{a}}(z'^* z)}{\sqrt{{}_\nu^{\mathbf{b}}\mathcal{N}_\phi^{\mathbf{a}}(|z|^2)\,{}_\nu^{\mathbf{b}}\mathcal{N}_\phi^{\mathbf{a}}(|z'|^2)}}. \end{aligned} \tag{13}$$

It follows that the GBHSs are normalized, but not orthogonal. The scalar product is well defined if the involved generalized basic hypergeometric functions converge. Throughout, we suppose that $0 < q < 1$. Let

$$u_l = (q^{-\nu})^{l(l+1)/2} \frac{(\mathbf{a}; q)_l}{(q;q)_l(\mathbf{b};q)_l}. \tag{14}$$

From

$$\frac{u_{l+1}}{u_l} = (q^{-\nu})^{l+1} \frac{1-a_{m+1}q^l}{1-q^{l+1}} \prod_{k=1}^{m} \left(\frac{1-a_k q^l}{1-b_k q^l}\right) \tag{15}$$

we deduce that the function ${}_\nu^{\mathbf{b}}\mathcal{N}_\phi^{\mathbf{a}}(x)$ is convergent everywhere on the positive axis if $\nu < 0$, implying that $|\mathbf{a}, \mathbf{b}; z\rangle_\phi^\nu$ is well defined everywhere on the complex plane and on the open unit disc if $\nu = 0$.

These states depend analytically on the complex variable z and thus belong to the very large class of holomorphic quantum states.

Let us end this section specifying the constraints on the parameters $a_1, \ldots, a_{m+1}$ and $b_1, \ldots, b_m$.

First, we suppose that $b_i \neq q^{-l}$, $i = 1, 2, \ldots, m$ and $l \in \mathbb{N}$; otherwise, the normalization function would be undefined. Moreover, to avoid undefined ${}^{\mathbf{b}}\rho_\phi^{\mathbf{a}}(l)$, we suppose that $a_i \neq q^{-l}$, $i = 1, 2, \ldots, m+1$ and $l \in \mathbb{N}$.

The positivity condition of the parameter functions ${}^{\mathbf{b}}\rho_\phi^{\mathbf{a}}(l)$ imposes the relations

$$\frac{1-q^{l+1}}{1-a_{m+1}q^l} \prod_{k=1}^{m} \left(\frac{1-a_k q^l}{1-b_k q^l}\right) > 0 \qquad \forall\, l = 0, 1, 2, \ldots, \tag{16}$$

which follows from the recurrence relation

$$\frac{{}^{\mathbf{b}}\rho_\phi^{\mathbf{a}}(l+1)}{{}^{\mathbf{b}}\rho_\phi^{\mathbf{a}}(l)} = \frac{1-q^{l+1}}{1-a_{m+1}q^l}\prod_{k=1}^{m}\left(\frac{1-a_kq^l}{1-b_kq^l}\right)\phi, \tag{17}$$

with the ground value ${}^{\mathbf{b}}\rho_\phi^{\mathbf{a}}(0) = 1$.

3. Generalized basic hypergeometric coherent states

In this section, we investigate generalized basic hypergeometric states GBHSs as an (over)complete set of states allowing a resolution of unity with a non-negative weight function.

The resolution of unity of the GBHSs assumes the existence of a positive weight function ${}^{\mathbf{b}}_{\nu}\mathcal{W}_\phi^{\mathbf{a}}(|z|^2)$ such that

$$\frac{1}{\pi}\iint \mathrm{d}^2z|\mathbf{a},\mathbf{b};z\rangle_\phi^\nu\ {}^{\mathbf{b}}_{\nu}\mathcal{W}_\phi^{\mathbf{a}}(|z|^2)\ {}^{\nu}_{\phi}\langle\mathbf{a},\mathbf{b};z| = \sum_{l=0}^{+\infty}|l\rangle\langle l| \equiv I, \tag{18}$$

where $\mathrm{d}^2z = \mathrm{d}(\mathrm{Re}\,z)\,\mathrm{d}(\mathrm{Im}\,z)$; the integration is over the complex plane or unit disc.

By substituting $z = r\,\mathrm{e}^{\mathrm{i}\varphi}$ into the left-hand side of (18) and integrating over φ, the function ${}^{\mathbf{b}}_{\nu}\mathcal{W}_\phi^{\mathbf{a}}(|z|^2)$ takes the form

$${}^{\mathbf{b}}_{\nu}\mathcal{W}_\phi^{\mathbf{a}}(x) = {}^{\mathbf{b}}_{\nu}\mathcal{N}_\phi^{\mathbf{a}}(x){}^{\mathbf{b}}_{\nu}\tilde{\mathcal{W}}_\phi^{\mathbf{a}}(x) \qquad x = r^2, \tag{19}$$

where ${}^{\mathbf{b}}_{\nu}\tilde{\mathcal{W}}_\phi^{\mathbf{a}}(x)$ has to be determined from the equation

$$\int_0^R x^l\ {}^{\mathbf{b}}_{\nu}\tilde{\mathcal{W}}_\phi^{\mathbf{a}}(x)\,\mathrm{d}x = q^{\nu l(l+1)/2}\ {}^{\mathbf{b}}\rho_\phi^{\mathbf{a}}(l) \qquad l = 0,1,2,\ldots. \tag{20}$$

The above problem is nothing but the moment power problem. If R is a finite quantity, then it reduces to the Haussdorff moment problem, while if R is infinite, then it becomes the Stieltjes moment problem [41].

If l in (20) is extended to $s-1$, where $s \in \mathbb{C}$, then the problem can be formulated in terms of the Mellin transform [42] which has been extensively used in the context of various kinds of generalized coherent states. Here $q^{\nu s(s-1)/2}\ {}^{\mathbf{b}}\rho_\phi^{\mathbf{a}}(s-1)$ is the Mellin transform, $\mathcal{M}\big[{}^{\mathbf{b}}_{\nu}\tilde{\mathcal{W}}_\phi^{\mathbf{a}}(x);s\big]$, of ${}^{\mathbf{b}}_{\nu}\tilde{\mathcal{W}}_\phi^{\mathbf{a}}(x)$, i.e.,

$$q^{\nu s(s-1)/2}\ {}^{\mathbf{b}}\rho_\phi^{\mathbf{a}}(s-1) = \mathcal{M}\big[{}^{\mathbf{b}}_{\nu}\tilde{\mathcal{W}}_\phi^{\mathbf{a}}(x);s\big] \equiv \int_0^R x^{s-1}\ {}^{\mathbf{b}}_{\nu}\tilde{\mathcal{W}}_\phi^{\mathbf{a}}(x)\,\mathrm{d}x. \tag{21}$$

As a matter of concrete realization, the states $|-,-;z\rangle_\phi^{-1}$ have the weight function [4]

$${}^{-}_{-1}\tilde{\mathcal{W}}_\phi^{-}(x) = \frac{1}{\phi(q)\ln(q^{-1})}\frac{1}{(-x/\phi;q)_\infty} \tag{22}$$

and the normalization function

$${}^{-}_{-1}\mathcal{N}_\phi^{-}(x) = \sum_{l=0}^{+\infty}\frac{q^{l(l+1)/2}}{(q;q)_l}\left(\frac{x}{\phi(q)}\right)^l. \tag{23}$$

The resolution of unity (18) can be used to introduce a new basis representation by sandwiching it between two arbitrary state vectors, $\langle\varphi|$ and $|\psi\rangle$, and writing the resulting scalar product in the form

$$\langle\varphi|\psi\rangle = \frac{1}{\pi}\iint \mathrm{d}^2z\,\big({}^{\mathbf{b}}_{\nu}\varphi_\phi^{\mathbf{a}}(z)\big)^*\ {}^{\mathbf{b}}_{\nu}\psi_\phi^{\mathbf{a}}(z) \tag{24}$$

with the wavefunctions, in the basis of generalized basic hypergeometric coherent states (GBHCSs), expressed as

$$
\begin{aligned}
{}^{\mathbf{b}}_{\nu}\varphi^{\mathbf{a}}_{\phi}(z) &= \sqrt{{}^{\mathbf{b}}_{\nu}\mathcal{W}^{\mathbf{a}}_{\phi}(|z|^2)}\; {}^{\nu}_{\phi}\langle \mathbf{a}, \mathbf{b}; z|\varphi\rangle \\
&= \left[\frac{{}^{\mathbf{b}}_{\nu}\mathcal{W}^{\mathbf{a}}_{\phi}(|z|^2)}{{}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}(|z|^2)}\right]^{1/2} \sum_{l=0}^{+\infty} (q^{-\nu})^{l(l+1)/4} \frac{z^{*l}}{\sqrt{{}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(l)}} \langle l|\varphi\rangle.
\end{aligned}
\tag{25}
$$

This defines the GBHCSs representation of the state $|\varphi\rangle$. The series appearing in (25) defines an entire analytic function of $\zeta = z^*$, which may be regarded as another basis representation. We call it the generalized basic hypergeometric analytic representation of $|\varphi\rangle$:

$$
{}^{\mathbf{b}}_{\nu}\tilde{\varphi}^{\mathbf{a}}_{\phi}(\zeta) = \sum_{l=0}^{+\infty} (q^{-\nu})^{l(l+1)/4} \frac{\zeta^{l}}{\sqrt{{}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(l)}} \langle l|\varphi\rangle.
\tag{26}
$$

The scalar product (24) can be then rewritten as

$$
\langle \varphi|\psi\rangle = \iint \mathrm{d}\mu(\zeta) \left({}^{\mathbf{b}}_{\nu}\tilde{\varphi}^{\mathbf{a}}_{\phi}(\zeta)\right)^{*} \; {}^{\mathbf{b}}_{\nu}\tilde{\psi}^{\mathbf{a}}_{\phi}(\zeta)
\tag{27}
$$

with the measure given by

$$
\mathrm{d}\mu(\zeta) = \frac{\mathrm{d}^2\zeta}{\pi} \frac{{}^{\mathbf{b}}_{\nu}\mathcal{W}^{\mathbf{a}}_{\phi}(|\zeta|^2)}{{}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}(|\zeta|^2)}.
\tag{28}
$$

Note that the basis analytic representations and the corresponding measure explicitly depend on the parameter set $\mathbf{a} = (a_1, \ldots, a_{m+1})$ and $\mathbf{b} = (b_1, \ldots, b_m)$. For a given state $|z\rangle$, many infinite basis analytic representations on the plane and unit disc can be defined, respectively, as elements of bases of Bargmann and Hardy spaces with measure (28).

Let us now investigate the self-reproducing property. The density matrix admits the expansion

$$
\sigma = \sum_{k,l=0}^{+\infty} \sigma(k,l)|l\rangle\langle k|
\tag{29}
$$

in Fock spaces. Between two GBHCSs, the matrix elements are expressed as

$$
\begin{aligned}
{}^{\nu}_{\phi}\langle \mathbf{a}, \mathbf{b}; z'|\sigma|\mathbf{a}, \mathbf{b}; z\rangle^{\nu}_{\phi} &= \left\{{}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}(|z|^2){}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}(|z'|^2)\right\}^{-1/2} \\
&\times \sum_{k,l=0}^{+\infty} \sigma(k,l)(q^{-\nu})^{l(l+1)/4}(q^{-\nu})^{k(k+1)/4} \frac{z'^{*k} z^{l}}{\sqrt{{}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(k)\, {}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(l)}} \equiv {}^{\mathbf{b}}_{\nu}\sigma^{\mathbf{a}}_{\phi}(z', z).
\end{aligned}
\tag{30}
$$

Using the completeness relation (18) for the GBHCSs, we write

$$
{}^{\mathbf{b}}_{\nu}\sigma^{\mathbf{a}}_{\phi}(z', z) = \int \mathrm{d}^2\zeta \; {}^{\mathbf{b}}_{\nu}K^{\mathbf{a}}_{\phi}(\zeta, z)\; {}^{\mathbf{b}}_{\nu}\sigma^{\mathbf{a}}_{\phi}(z', \zeta),
\tag{31}
$$

where the reproducing kernel ${}^{\mathbf{b}}_{\nu}K^{\mathbf{a}}_{\phi}(\zeta, z)$ is defined as

$$
{}^{\mathbf{b}}_{\nu}K^{\mathbf{a}}_{\phi}(\zeta, z) = \frac{1}{\pi}\langle \zeta|\mathbf{a}, \mathbf{b}; z\rangle^{\nu}_{\phi}.
\tag{32}
$$

Equation (32) gives the self-reproducing property of ${}^{\mathbf{b}}_{\nu}\sigma^{\mathbf{a}}_{\phi}(z', z)$ with ${}^{\mathbf{b}}_{\nu}K^{\mathbf{a}}_{\phi}(\zeta, z)$ as the self-reproducing kernel. The kernel satisfies the required properties of a self-reproducing kernel. Namely, firstly, it satisfies the matrix multiplication property:

$$
\int \mathrm{d}^2\zeta \; {}^{\mathbf{b}}_{\nu}K^{\mathbf{a}}_{\phi}(z, \zeta){}^{\mathbf{b}}_{\nu}K^{\mathbf{a}}_{\phi}(\zeta, z') = {}^{\mathbf{b}}_{\nu}K^{\mathbf{a}}_{\phi}(z, z').
\tag{33}
$$

Secondly, since σ is Hermitian, the kernel satisfies the Hermicity property:

$$\left({}^{\mathbf{b}}_{\nu}K^{\mathbf{a}}_{\phi}(x,\zeta)\right)^{*} = {}^{\mathbf{b}}_{\nu}K^{\mathbf{a}}_{\phi}(\zeta,x). \tag{34}$$

From (13), it follows that ${}^{\mathbf{b}}_{\nu}K^{\mathbf{a}}_{\phi}(z,z) > 0$.

4. Generalized basic hypergeometric states as eigenstates of lowering operators

In this section, we define raising and lowering operators. The GBHSs, defined above in section 3, are eigenstates of the lowering operators. Up to a factor, these operators appear as a deformation of the creation and annihilation operators of the non-deformed harmonic oscillator.

The generalized basic hypergeometric lowering and raising operators are defined by

$${}^{\mathbf{b}}_{\nu}\mathcal{U}^{\mathbf{a}}_{\phi} = \sum_{l=0}^{+\infty} {}^{\mathbf{b}}_{\nu}\mathcal{F}^{\mathbf{a}}_{\phi}(l)|l\rangle\langle l+1| \tag{35a}$$

$$\left({}^{\mathbf{b}}_{\nu}\mathcal{U}^{\mathbf{a}}_{\phi}\right)^{\dagger} = \sum_{l=0}^{+\infty} {}^{\mathbf{b}}_{\nu}\mathcal{F}^{\mathbf{a}}_{\phi}(l)|l+1\rangle\langle l|, \tag{35b}$$

where the coefficients ${}^{\mathbf{b}}_{\nu}\mathcal{F}^{\mathbf{a}}_{\phi}(l)$ are given by

$${}^{\mathbf{b}}_{\nu}\mathcal{F}^{\mathbf{a}}_{\phi}(l) = (q^{\nu})^{(l+1)/2}\sqrt{\frac{1-q^{l+1}}{1-a_{m+1}q^{l}}\prod_{k=1}^{m}\left(\frac{1-b_k q^{l}}{1-a_k q^{l}}\right)\phi}. \tag{36}$$

From (16), one can observe that these coefficients are real and positive. The ladder operators obey the noncanonical commutation relation,

$$\left[{}^{\mathbf{b}}_{\nu}\mathcal{U}^{\mathbf{a}}_{\phi}, \left({}^{\mathbf{b}}_{\nu}\mathcal{U}^{\mathbf{a}}_{\phi}\right)^{\dagger}\right] = \sum_{l=0}^{+\infty}\left[\left({}^{\mathbf{b}}_{\nu}\mathcal{F}^{\mathbf{a}}_{\phi}(l)\right)^2 - \left({}^{\mathbf{b}}_{\nu}\mathcal{F}^{\mathbf{a}}_{\phi}(l-1)\right)^2\right]|l\rangle\langle l|, \tag{37}$$

where ${}^{\mathbf{b}}_{\nu}\mathcal{F}^{\mathbf{a}}_{\phi}(-1) \equiv 0$ by definition. They act on the Fock space basis as follows:

$${}^{\mathbf{b}}_{\nu}\mathcal{U}^{\mathbf{a}}_{\phi}|l\rangle = {}^{\mathbf{b}}_{\nu}\mathcal{F}^{\mathbf{a}}_{\phi}(l-1)|l-1\rangle, \tag{38a}$$

$$\left({}^{\mathbf{b}}_{\nu}\mathcal{U}^{\mathbf{a}}_{\phi}\right)^{\dagger}|l\rangle = {}^{\mathbf{b}}_{\nu}\mathcal{F}^{\mathbf{a}}_{\phi}(l)|l+1\rangle \tag{38b}$$

that justifies their names of lowering and raising operators, respectively. It is then easy to show that the GBHSs are eigenstates of the lowering operator:

$${}^{\mathbf{b}}_{\nu}\mathcal{U}^{\mathbf{a}}_{\phi}|\mathbf{a},\mathbf{b};z\rangle^{\nu}_{\phi} = z|\mathbf{a},\mathbf{b};z\rangle^{\nu}_{\phi} \tag{39}$$

with the complex eigenvalue z. The link between the coefficients ${}^{\mathbf{b}}\mathcal{F}^{\mathbf{a}}_{\phi}(l)$ defining the lowering operator and the parameter function ${}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(l)$ is given by

$${}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(l) = (q^{-\nu})^{l(l+1)/2}\left({}^{\mathbf{b}}_{\nu}\mathcal{F}^{\mathbf{a}}_{\phi}(0)\,{}^{\mathbf{b}}_{\nu}\mathcal{F}^{\mathbf{a}}_{\phi}(1)\ldots{}^{\mathbf{b}}_{\nu}\mathcal{F}^{\mathbf{a}}_{\phi}(l-1)\right)^2. \tag{40}$$

From (39), we deduce the relation

$$\frac{1}{\sqrt{1-q}}\,{}^{-}_{\nu}\mathcal{U}^{-}_{\phi}|-,-;(1-q)^{1/2}z\rangle^{\nu}_{\phi} = z|-,-;(1-q)^{1/2}z\rangle^{\nu}_{\phi}, \tag{41}$$

which reduces to

$$a|z\rangle = z|z\rangle \tag{42}$$

with

$$a = \sum_{l=0}^{+\infty} \sqrt{l+1}|l\rangle\langle l+1|, \tag{43}$$

as $q \to 1$. It is then natural to call ${}^{\mathbf{b}}_{\nu}\mathcal{A}^{\mathbf{a}}_{\phi} = \frac{1}{\sqrt{1-q}}\, {}^{\mathbf{b}}_{\nu}\mathcal{U}^{\mathbf{a}}_{\phi}$(resp. $\left({}^{\mathbf{b}}_{\nu}\mathcal{A}^{\mathbf{a}}_{\phi}\right)^{\dagger} = \frac{1}{\sqrt{1-q}}\left({}^{\mathbf{b}}_{\nu}\mathcal{U}^{\mathbf{a}}_{\phi}\right)^{\dagger}$) the generalized basic hypergeometric annihilation (resp. creation) operators. Given such non-Hermitian operators, it is worthy of defining relevant Hermitian combinations by

$$ {}^{\mathbf{b}}_{\nu}\mathcal{Q}^{\mathbf{a}}_{\phi} = \frac{1}{\sqrt{2}}\left[\left({}^{\mathbf{b}}_{\nu}\mathcal{A}^{\mathbf{a}}_{\phi}\right)^{\dagger} + {}^{\mathbf{b}}_{\nu}\mathcal{A}^{\mathbf{a}}_{\phi}\right] \tag{44a}$$

$$ {}^{\mathbf{b}}_{\nu}\mathcal{P}^{\mathbf{a}}_{\phi} = \frac{\mathrm{i}}{\sqrt{2}}\left[\left({}^{\mathbf{b}}_{\nu}\mathcal{A}^{\mathbf{a}}_{\phi}\right)^{\dagger} - {}^{\mathbf{b}}_{\nu}\mathcal{A}^{\mathbf{a}}_{\phi}\right], \tag{44b}$$

which can be interpreted as generalized basic hypergeometric (momentum) coordinate operators.

Let us now present a realization of the algebra spanned by the operators ${}^{\mathbf{b}}_{\nu}\mathcal{A}^{\mathbf{a}}_{\phi}$ and $\left({}^{\mathbf{b}}_{\nu}\mathcal{A}^{\mathbf{a}}_{\phi}\right)^{\dagger}$ for particular cases of parameters $\mathbf{a}$, $\mathbf{b}$, ν and ϕ:

(i) Case $\mathbf{a} = -$, $\mathbf{b} = -$.

Let

$$[j]^{\nu}_{q} = q^{\nu j}\frac{1-q^{j}}{1-q}\phi(q). \tag{45}$$

Then, from the relations

$$q^{-\nu}[j+1]^{\nu}_{q} - [j]^{\nu}_{q} = q^{j(\nu+1)}\phi(q), \qquad q^{-\nu}[j+1]^{\nu}_{q} - q[j]^{\nu}_{q} = q^{\nu j}\phi(q),$$

we readily obtain

$$q^{-\nu}\, {}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\left({}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\right)^{\dagger} - \left({}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\right)^{\dagger}\, {}^{-}_{\nu}\mathcal{A}^{-}_{\phi} = \phi(q)\sum_{j=0}^{+\infty} q^{j(\nu+1)}|j\rangle\langle j|, \tag{46a}$$

$$q^{-\nu}\, {}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\left({}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\right)^{\dagger} - q\left({}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\right)^{\dagger}\, {}^{-}_{\nu}\mathcal{A}^{-}_{\phi} = \phi(q)\sum_{j=0}^{+\infty} q^{j\nu}|j\rangle\langle j|. \tag{46b}$$

Setting $N = a^{\dagger}a$, the number operator of the non-deformed oscillator algebra, leads to

$$q^{-\nu}\, {}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\left({}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\right)^{\dagger} - \left({}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\right)^{\dagger}\, {}^{-}_{\nu}\mathcal{A}^{-}_{\phi} = q^{N(\nu+1)}\phi(q) \tag{47a}$$

$$q^{-\nu}\, {}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\left({}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\right)^{\dagger} - q\left({}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\right)^{\dagger}\, {}^{-}_{\nu}\mathcal{A}^{-}_{\phi} = q^{N\nu}\phi(q) \tag{47b}$$

$$\left[{}^{-}_{\nu}\mathcal{A}^{-}_{\phi}, N\right] = {}^{-}_{\nu}\mathcal{A}^{-}_{\phi} \qquad \left[N, \left({}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\right)^{\dagger}\right] = \left({}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\right)^{\dagger}. \tag{47c}$$

On the space Γ of all finite combinations of the monomials z^{n}, $z \in \mathbb{C}$, $n \in \mathbb{Z}$:

$$\Gamma = \left\{\sum_{n\in P} a_{n}z^{n}, a_{n} \in \mathbb{C}, P \subset \mathbb{Z}\right\},$$

we can perform a realization of the algebra (47a)–(47c) in the following terms:

$$\left({}^{-}_{\nu}\mathcal{A}^{-}_{\phi}\right)^{\dagger} h(z) := zh(z) \tag{48a}$$

$$ {}_{\nu}^{-}\mathcal{A}_{\phi}^{-}h(z) := \frac{\phi(q)}{z(1-q)}q^{\nu\rho}[h(q^{\nu}z) - q^{\rho}h(q^{\nu+1}z)] \tag{48b} $$

$$ Nh(z) := \left(\rho + z\frac{\mathrm{d}}{\mathrm{d}z}\right)h(z). \tag{48c} $$

(ii) Case $\nu = 0$, $\phi(q) = 1$, $a_{m+1} = a$ and $a_k = b_k = 0$ for $0 \leqslant k \leqslant m$.
In this case, denoting the operators ${}_{\nu}^{\mathbf{b}}\mathcal{A}_{\phi}^{\mathbf{a}}$ and $\left({}_{\nu}^{\mathbf{b}}\mathcal{A}_{\phi}^{\mathbf{a}}\right)^{\dagger}$ by A and $A^{\dagger}$, respectively, one can show that

$$ AA^{\dagger}|n\rangle = q_{n+1}|n\rangle, \qquad A^{\dagger}A|n\rangle = q_n|n\rangle, \tag{49} $$

where

$$ q_n = \frac{1}{1-q}\frac{1-q^n}{1-aq^{n-1}}. \tag{50} $$

Since $A^{\dagger}A$ and $AA^{\dagger}$ are diagonalized in the Fock space $\{|n\rangle, n = 0, 1, 2, \ldots\}$, any relation $\mathcal{R}(q_{n+1}, q_n) = 0$ is also valid for the operators [38]:

$$ \mathcal{R}(A^{\dagger}A, AA^{\dagger}) = 0. \tag{51} $$

Hence, one can straightforwardly establish the relation

$$ a(1-q)^2AA^{\dagger}A^{\dagger}A + qI = (a-q^2)A^{\dagger}A + q(1-a)AA^{\dagger}. \tag{52} $$

Then, the following statement holds.

Proposition 4.1. *On the space $\mathcal{O}(D_R)$ of holomorpic functions on the disc D_R of radius R, the algebra (52) admits the following realization:*

$$ A^{\dagger}\varphi(z) := z\varphi(z) \tag{53a} $$

$$ A\varphi(z) := \frac{r}{z}\left\{\left(\frac{\alpha-1}{1-\alpha Q}\right)\varphi(z) + \varphi(z)\right\}, \tag{53b} $$

where

$$ \alpha = \frac{a}{q}, \qquad r = \frac{1}{\alpha(1-q)}, \tag{54} $$

and the operator Q acts as $Q\psi(z) = \psi(qz)$, for all $\psi \in \mathcal{O}(D_R)$.

Proof. First, note that by using the identities (54), (52) can be rewritten as

$$ (1-q)AA^{\dagger}A^{\dagger}A + rI = r(\alpha - q)A^{\dagger}A + r(1-\alpha q)AA^{\dagger}. \tag{55} $$

From the relations

$$ A^{\dagger}A\varphi(z) = r\left\{\left(\frac{\alpha-1}{1-\alpha Q}\right)\varphi(z) + \varphi(z)\right\} $$

$$ AA^{\dagger}\varphi(z) = r\left\{\left(\frac{\alpha-1}{1-\alpha qQ}\right)\varphi(z) + \varphi(z)\right\} $$

and the decomposition

$$ \frac{1}{(1-\alpha Q)(1-\alpha qQ)} = \frac{1}{1-q}\left(\frac{1}{1-\alpha Q} - \frac{q}{1-\alpha qQ}\right) $$

we have, for any $\varphi \in \mathcal{O}(D_R)$,

$$
\begin{aligned}
(1-q)AA^\dagger A^\dagger A\varphi(z) + r\varphi(z) &= r(\alpha-1)(1-q)AA^\dagger \frac{1}{1-\alpha q Q}\varphi(z) \\
&\quad + r^2(1-q)\left(\frac{\alpha-1}{1-\alpha q Q}\varphi(z)+\varphi(z)\right) + r\varphi(z) \\
&= \frac{r^2(1-q)(\alpha-1)^2}{(1-\alpha q Q)(1-\alpha Q)}\varphi(z) + \frac{r^2(1-q)(\alpha-1)}{1-\alpha Q}\varphi(z) \\
&\quad + \frac{r^2(1-q)(\alpha-1)}{1-\alpha q Q}\varphi(z) + r^2(1-q)\varphi(z) + r\varphi(z) \\
&= r^2(\alpha-q)\left(\frac{\alpha-1}{1-\alpha Q}\varphi(z)+\varphi(z)\right) + r^2(1-\alpha q)\left(\frac{\alpha-1}{1-\alpha q Q}\varphi(z)+\varphi(z)\right) \\
&\quad - r(\alpha-q)\varphi(z) - r^2(1-\alpha q)\varphi(z) + r\varphi(z) + r^2(1-q)\varphi(z) \\
&= r(\alpha-q)A^\dagger A\varphi(z) + r(1-\alpha q)AA^\dagger\varphi(z).
\end{aligned}
$$

□

When $a = 0$, the algebra (52) reduces to

$$AA^\dagger - qA^\dagger A = I, \tag{56}$$

which defines nothing but the standard q-deformation of the harmonic oscillator [43].

5. Husimi and Husimi phase distributions

5.1. Husimi distribution

The Husimi distribution for the generalized basic hypergeometric states $|\mathbf{a}, \mathbf{b}; z\rangle_\phi^\nu$ is given by the squared of the modulus of the wavefunction in the conventional coherent states $|\alpha\rangle$:

$$\mathcal{Q}_{|\mathbf{a},\mathbf{b};z\rangle_\phi^\nu} = \frac{1}{2\pi}|\langle\alpha|\mathbf{a}, \mathbf{b}; z\rangle_\phi^\nu|^2 \tag{57}$$

with

$$\langle\alpha|\mathbf{a}, \mathbf{b}; z\rangle_\phi^\nu = \left[\frac{\mathrm{e}^{-|\alpha|^2}}{{}_\nu^{\mathbf{b}}\mathcal{N}_\phi^{\mathbf{a}}(|z|^2)}\right]^{1/2} \sum_{l=0}^{+\infty} (q^{-\nu})^{l(l+1)/4} \frac{(\alpha^* z)^l}{\sqrt{l!\,{}^{\mathbf{b}}\rho_\phi^{\mathbf{a}}(l)}}. \tag{58}$$

$\mathcal{Q}_{|\mathbf{a},\mathbf{b};z\rangle_\phi^\nu}$ is positive, normalized to 1 with measure $\mathrm{d}^2\alpha$ and yields a two-dimensional probability distribution over the complex α-plane.

By letting $\alpha = |\alpha|\,\mathrm{e}^{\mathrm{i}\theta}$, the Husimi phase distribution is defined by

$$\mathcal{P}_{|\mathbf{a},\mathbf{b};z\rangle_\phi^\nu}(\theta) := \int_0^{+\infty} \mathcal{Q}_{|\mathbf{a},\mathbf{b};z\rangle_\phi^\nu}\, \mathrm{d}|\alpha|^2. \tag{59}$$

Since

$$\left|\langle\alpha|\mathbf{a}, \mathbf{b}; z\rangle_\phi^\nu\right|^2 = \frac{\mathrm{e}^{-|\alpha|^2}}{{}_\nu^{\mathbf{b}}\mathcal{N}_\phi^{\mathbf{a}}(|z|^2)} \sum_{k,l=0}^{+\infty} \frac{z^{*k} z^l |\alpha|^{k+l}\, \mathrm{e}^{\mathrm{i}(k-l)\theta}}{\sqrt{l!k!\,{}^{\mathbf{b}}\rho_\phi^{\mathbf{a}}(l)\,{}^{\mathbf{b}}\rho_\phi^{\mathbf{a}}(k)}} (q^{-\nu})^{l(l+1)/4}(q^{-\nu})^{k(k+1)/4}, \tag{60}$$

we obtain

$$\mathcal{P}_{|\mathbf{a},\mathbf{b};z\rangle_\phi^\nu}(\theta) = \frac{1}{2\pi}\sum_{k,l=0}^{+\infty} \langle l|\mathbf{a}, \mathbf{b}; z\rangle_\phi^\nu \left(\langle k|\mathbf{a}, \mathbf{b}; z\rangle_\phi^\nu\right)^* \mathcal{G}(l,k)\, \mathrm{e}^{-\mathrm{i}(l-k)\theta} \tag{61}$$

with

$$\mathcal{G}(l,k) = \frac{\Gamma\left(\frac{k+l}{2}+1\right)}{\sqrt{\Gamma(l+1)\Gamma(k+1)}}. \tag{62}$$

One can readily check that $\mathcal{Q}_{|-,-;(1-q)^{1/2}z\rangle_\phi^\nu}$ reduces to the conventional Husimi distribution $(|\langle\alpha|z\rangle|^2)/2\pi$ with

$$|\langle\alpha|z\rangle| = \mathrm{e}^{-(|z|^2+|\alpha|^2)/2}\sum_{l=0}^{+\infty}\frac{(\alpha^* z)^l}{l!} \tag{63}$$

as $q \to 1$.

5.2. Generalized basic hypergeometric Husimi and Husimi phase distributions

By analogy with the conventional Husimi distribution, we now define the generalized basic hypergeometric Husimi distribution of normalized states $|\psi\rangle$ by the square of modulus of the wavefunction in the GBHCSs basis, $\psi(z) := \sqrt{{}^{\mathbf{b}}_{\nu}\mathcal{W}^{\mathbf{a}}_{\phi}(|z|^2)}\;{}^{\nu}_{\phi}\langle\mathbf{a},\mathbf{b};z|\psi\rangle$, by

$$\mathcal{Q}_{|\psi\rangle}(z) = \frac{1}{\pi}\,{}^{\mathbf{b}}_{\nu}\mathcal{W}^{\mathbf{a}}_{\phi}(|z|^2)\;{}^{\nu}_{\phi}\langle\mathbf{a},\mathbf{b};z|\psi\rangle\langle\psi|\mathbf{a},\mathbf{b};z\rangle^{\nu}_{\phi}, \tag{64}$$

which provides a two-dimensional true probability distribution over the complex z-plane for $\nu < 0$ and over the unit disc for $\nu = 0$. As one may check, the normalization condition is due to the resolution of unity. It is therefore mandatory for the states $|\mathbf{a},\mathbf{b};z\rangle^{\nu}_{\phi}$ to be coherent.

Defining generalized basic hypergeometric Husimi distributions, we formally replace the coherent states in the usual Husimi distribution by the GHBCSs and introduce the weight function into the distribution rather than into the integration measure.

The generalized basic hypergeometric Husimi distributions can be used to define corresponding phase distributions by integrating over the modulus of the complex variable $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}\,(x = |z|^2)$:

$$\mathcal{P}_{|\psi\rangle}(\theta) := \frac{1}{2\pi}\int_0^R {}^{\mathbf{b}}_{\nu}\mathcal{W}^{\mathbf{a}}_{\phi}(|z|^2)\;{}^{\nu}_{\phi}\langle\mathbf{a},\mathbf{b};z|\psi\rangle\langle\psi|\mathbf{a},\mathbf{b};z\rangle^{\nu}_{\phi}\,\mathrm{d}|z|^2. \tag{65}$$

By letting $\psi_l = \langle l|\psi\rangle$ and making use of (21), we obtain

$$\mathcal{P}_{|\psi\rangle}(\theta) = \frac{1}{2\pi}\sum_{k,l=0}^{+\infty}\left(q^{-\nu(k-l)^2/8}\right){}^{\mathbf{b}}\mathcal{G}^{\mathbf{a}}_{\phi}(l,k)\psi_l^*\psi_k\,\mathrm{e}^{-\mathrm{i}(k-l)\theta} \tag{66}$$

with

$${}^{\mathbf{b}}\mathcal{G}^{\mathbf{a}}_{\phi}(l,k) = \frac{{}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}\left(\frac{k+l}{2}\right)}{\sqrt{{}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(l)\,{}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(k)}}. \tag{67}$$

${}^{\mathbf{b}}\mathcal{G}^{\mathbf{a}}_{\phi}(l,k)$ are symmetric in l and k and normalized to 1 for $l = k$, thus ensuring the normalization condition

$$\int \mathrm{d}\theta\,\mathcal{P}_{|\psi\rangle}(\theta) = 1. \tag{68}$$

For Fock states $|N\rangle$, we have $\psi_l = \delta_{l,N}$ and the phase distribution (66) reduces to

$$\mathcal{P}_{|N\rangle}(\theta) = \frac{1}{2\pi}. \tag{69}$$

6. Geometrical and quantum optical properties

In this section, we study some geometrical and physical properties of the GBHSs $|\mathbf{a},\mathbf{b};(1-q)^{1/2}z\rangle^{\nu}_{\phi}$.

6.1. Photon number distribution

The GBHSs, $|\mathbf{a},\mathbf{b};(1-q)^{1/2}z\rangle^{\nu}_{\phi}$, have the Fock representation,

$$\langle l|\mathbf{a},\mathbf{b};(1-q)^{1/2}z\rangle^{\nu}_{\phi} = \frac{(1-q)^{l/2}(q^{-\nu})^{l(l+1)/4}}{\sqrt{{}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)|z|^2)\,{}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(l)}}, \tag{70}$$

from which follows the probability of finding the state $|l\rangle$ in the state $|\mathbf{a},\mathbf{b};z\rangle^{\nu}_{\phi}$ (photon number distribution) expressed as

$$\tilde{\mathcal{P}}_{|\mathbf{a},\mathbf{b};(1-q)^{1/2}z\rangle^{\nu}_{\phi}}(l,x) = \frac{(1-q)^{l}(q^{-\nu})^{l(l+1)/2}}{{}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)|z|^2)\,{}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(l)}x^{l} \qquad x=|z|^2. \tag{71}$$

$\tilde{\mathcal{P}}_{|-,-;(1-q)^{1/2}z\rangle^{\nu}_{\phi}}(l,x)$ reduces to a Poisson distribution for the conventional CSs in the limit $q\to 1$.

The expectation values of the monomials $(a^{\dagger})^{r}a^{r}$, where a and $a^{\dagger}$ are the conventional boson annihilation and creation operators, are expressible through the derivatives of ${}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)x)$ as [32]

$$\begin{aligned}{}^{\nu}_{\phi}\langle\mathbf{a},\mathbf{b};z|(a^{\dagger})^{r}a^{r}|\mathbf{a},\mathbf{b};z\rangle^{\nu}_{\phi} &\equiv {}^{\nu}_{\phi}\langle(a^{\dagger})^{r}a^{r}\rangle^{\nu}_{\phi}\\ &= \frac{x^{r}}{{}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)x)}\frac{\mathrm{d}^{r}\,{}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)x)}{\mathrm{d}x^{r}} \qquad r=0,1,2,\ldots.\end{aligned} \tag{72}$$

A Poisson distribution is characterized by the fact that the variance of the number operator $N=a^{\dagger}a$ is equal to its average. The deviation from Poisson statistics can be measured with the Mandel parameter ${}^{\mathbf{b}}_{\nu}\mathcal{Q}^{\mathbf{a}}_{\phi}(x)$ [44]:

$${}^{\mathbf{b}}_{\nu}\mathcal{Q}^{\mathbf{a}}_{\phi}(x) = \frac{\left((\Delta N)^{\nu}_{\phi}\right)^2-\left({}^{\nu}_{\phi}\langle N\rangle^{\nu}_{\phi}\right)^2}{{}^{\nu}_{\phi}\langle N\rangle^{\nu}_{\phi}}, \qquad \left((\Delta N)^{\nu}_{\phi}\right)^2 = {}^{\nu}_{\phi}\langle N^2\rangle^{\nu}_{\phi}-\left({}^{\nu}_{\phi}\langle N\rangle^{\nu}_{\phi}\right)^2, \tag{73}$$

which vanishes for the Poisson distribution. By using (72) to evaluate the averages in (73), one easily obtains

$${}^{\mathbf{b}}_{\nu}\mathcal{Q}^{\mathbf{a}}_{\phi}(x) = x\left(\frac{\left({}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)x)\right)''}{\left({}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)x)\right)'}-\frac{\left({}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)x)\right)'}{{}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)x)}\right). \tag{74}$$

Here primes denote the orders of differentiation with respect to the variable x. Thus, the statistical properties of the states $|\mathbf{a},\mathbf{b};z\rangle^{\nu}_{\phi}$ only depend on the growth properties of the normalization function ${}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)x)$. A state for which ${}^{\mathbf{b}}_{\nu}\mathcal{Q}^{\mathbf{a}}_{\phi}(x)<0$ (resp. >0) is called sub-Poissonian (resp. super-Poissonian).

As illustrated in figure 2, ${}^{\mathbf{b}}_{-1}\mathcal{Q}^{\mathbf{a}}_{1}(x)<0$ yielding a sub-Poissonian distribution.

6.2. Metric factor

The correspondence from z to $|\mathbf{a},\mathbf{b};(1-q)^{1/2}z\rangle^{\nu}_{\phi}$ is a mapping from the space $\mathbb{C}$ of complex numbers into a continuous subset of unit vectors in the Hilbert space. As such, one may imagine that this map generates a two-dimensional (because $\mathbb{C}=\mathbb{R}\oplus\mathbb{R}$) surface sweeping

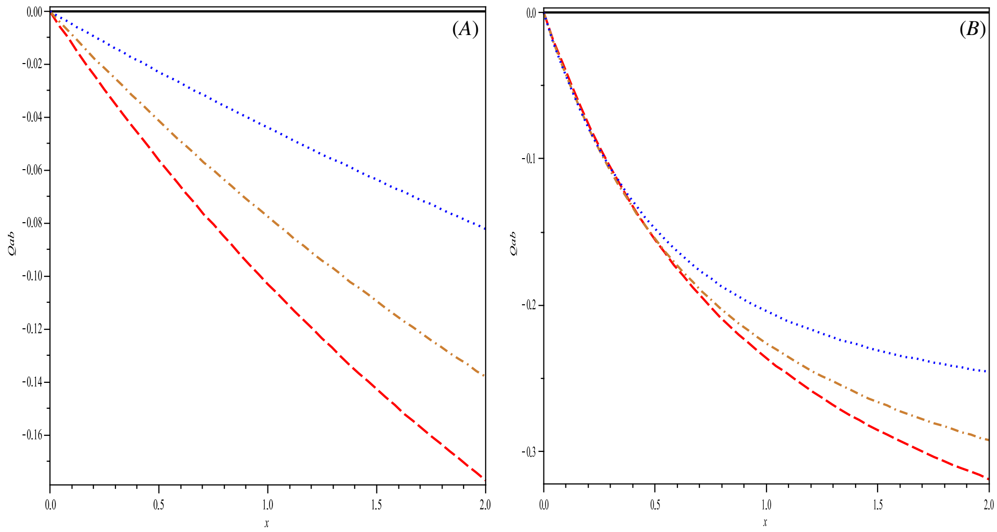

Figure 2. The Mandel parameter ${}^{\mathbf{b}}_{-1}\mathcal{Q}^{\mathbf{a}}_{1}(x) \equiv Qab(x)$ versus x for various q values: $q = 0.7$ (dash), $q = 0.8$ (dashdot), $q = 0.9$ (dot).

through an infinite-dimensional Hilbert space. The two-dimensional surface can be described by its geometry which, in an explicit form, is represented by the induced two-dimensional Riemannian metric tensor in the line element $\mathrm{d}\sigma^{\nu}_{\phi}$. This metric is not that induced directly by the Hilbert space metric itself, but rather one induced by the physical content of the Hilbert space in which vectors differing only in phase are identified. A suitable metric between any two Hilbert space vectors, say $|\psi\rangle$ and $|\phi\rangle$, is thus the ray metric defined by

$$\mathrm{d}_{\mathrm{ray}}(|\psi\rangle; |\phi\rangle) := \min_{0\leqslant\alpha<2\pi} \||\psi\rangle - \mathrm{e}^{\mathrm{i}\alpha}|\phi\rangle\|. \tag{75}$$

The infinitesimal form of this metric is given by the Fubini–Study metric which, restricted to coherent states, takes the form

$$\mathrm{d}\sigma := \left\|\mathrm{d}|\mathbf{a}, \mathbf{b}; z\rangle^{\nu}_{\phi}\right\|^2 - \left|{}^{\nu}_{\phi}\langle\mathbf{a}, \mathbf{b}; z|\, \mathrm{d}|\mathbf{a}, \mathbf{b}; z\rangle^{\nu}_{\phi}\right|^2. \tag{76}$$

A moment's reflection shows that

$$\mathrm{d}\sigma^{\nu}_{\phi} = {}^{\mathbf{b}}_{\nu}\mathcal{W}^{\mathbf{a}}_{\phi}(x)\, \mathrm{d}z^*\, \mathrm{d}z \tag{77}$$

with

$${}^{\mathbf{b}}_{\nu}\mathcal{W}^{\mathbf{a}}_{\phi}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\, {}^{\nu}_{\phi}\langle N\rangle^{\nu}_{\phi} = \left(\frac{x\left({}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)x)\right)'}{{}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)x)}\right)' \qquad x = |z|^2 \tag{78}$$

as the corresponding metric factor. Therefore, the result is a circular symmetric two-dimensional geometry. If ${}^{\mathbf{b}}_{\nu}\mathcal{W}^{\mathbf{a}}_{\phi}(x) \equiv 1, \mathrm{d}\sigma^{\nu}_{\phi}$ describes a flat two-dimensional surface. Otherwise, the geometry is non-flat.

Under the conditions on which the graphs are given, figure 3 shows that, for the GBHSs $|\mathbf{a}, \mathbf{b}; (1-q)^{1/2}z\rangle^{-1}_{1}$, the metric factor may be less or greater than unity according to the values of x. Therefore, the geometry is non-flat.

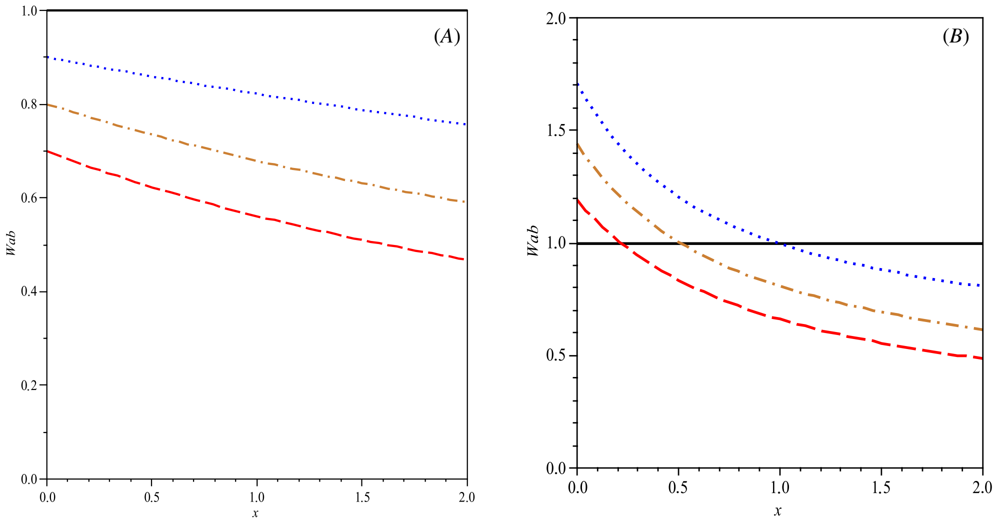

Figure 3. The metric factor ${}^{\mathbf{b}}_{-1}\mathcal{W}^{\mathbf{a}}_{1}(x) \equiv Wab(x)$ versus x for various q values: $q = 0.7$ (dash), $q = 0.8$ (dashdot), $q = 0.9$ (dot).

6.3. Squeezing properties

Let us consider the conventional quadrature operators X and P defined in terms of non-deformed operators a and $a^\dagger$:

$$X = \frac{1}{\sqrt{2}}(a + a^\dagger), \qquad P = \frac{1}{\mathrm{i}\sqrt{2}}(a - a^\dagger). \tag{79}$$

The commutation relation for a and $a^\dagger$ leads to the following uncertainty relation:

$$(\Delta X)^2(\Delta P)^2 \geqslant \tfrac{1}{4}|[X, P]|^2 = \tfrac{1}{4}. \tag{80}$$

In the vacuum state $|0\rangle$, we have $(\Delta X)^2_0 = 1/2$ and $(\Delta P)^2_0 = 1/2$, and so $(\Delta X)^2_0(\Delta P)^2_0 = 1/4$. While it is impossible to lower the product $(\Delta X)^2(\Delta P)^2$ below the vacuum uncertainty value, it is nevertheless possible to define the squeezed states for which at most one quadrature variance lies below the vacuum value:

$$(\Delta X)^2 < \tfrac{1}{2} \qquad \text{or} \qquad (\Delta P)^2 < \tfrac{1}{2}. \tag{81}$$

For the states $|\mathbf{a}, \mathbf{b}; (1-q)^{1/2}z\rangle^{\nu}_{\phi}$, it is straightforward to show that the variances of X and P are given by

$$(\Delta X)^2 = 2(\mathrm{Re}(z))^2\left[\mathbf{S}^{\nu(2,0)}_{\phi}(x) - \left(\mathbf{S}^{\nu(1,0)}_{\phi}(x)\right)^2\right] + x\left[\mathbf{S}^{\nu(1,1)}_{\phi}(x) - \mathbf{S}^{\nu(2,0)}_{\phi}(x)\right] + \tfrac{1}{2}, \tag{82}$$

$$(\Delta P)^2 = 2(\mathrm{Im}(z))^2\left[\mathbf{S}^{\nu(2,0)}_{\phi}(x) - \left(\mathbf{S}^{\nu(1,0)}_{\phi}(x)\right)^2\right] + x\left[\mathbf{S}^{\nu(1,1)}_{\phi}(x) - \mathbf{S}^{\nu(2,0)}_{\phi}(x)\right] + \tfrac{1}{2}, \tag{83}$$

where

$$\mathbf{S}^{\nu(p,r)}_{\phi}(x) = \frac{1}{{}^{\mathbf{b}}_{\nu}\mathcal{N}^{\mathbf{a}}_{\phi}((1-q)x)} \sum_{l=0}^{+\infty} (1-q)^{l+(p+r)/2} q^{-\nu(l+p)(l+p+1)/4} q^{-\nu(l+r)(l+r+1)/4}$$
$$\times \left(\frac{(l+r)!(l+p)!}{{}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(l+r)\,{}^{\mathbf{b}}\rho^{\mathbf{a}}_{\phi}(l+p)}\right)^{1/2} \frac{x^l}{l!} \qquad r, p = 0, 1, 2, \ldots. \tag{84}$$

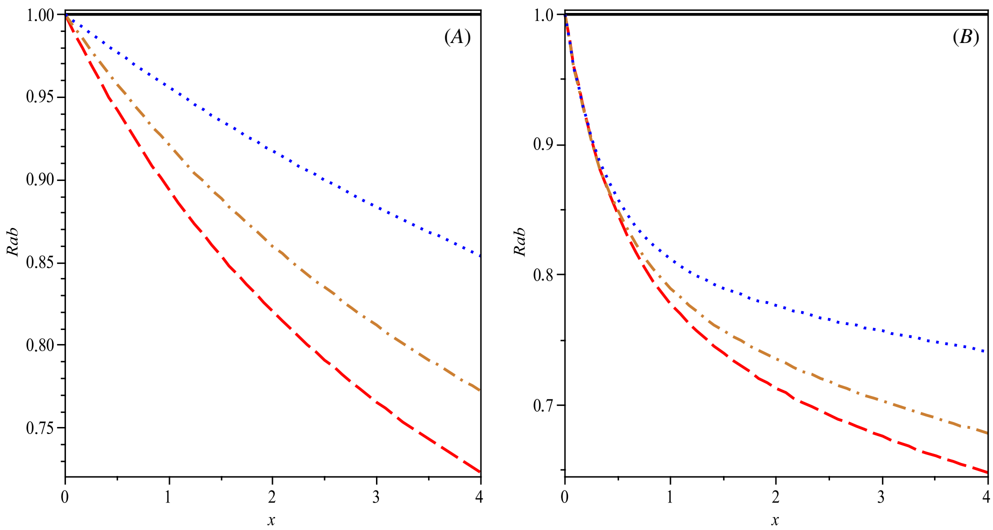

Figure 4. The variance ratio ${}^{\mathbf{b}}_{-1}\mathcal{R}^{\mathbf{a}}_{1}(x) \equiv Rab(x)$ versus x for various q values: $q = 0.7$ (dash), $q = 0.8$ (dashdot), $q = 0.9$ (dot).

The plot of ${}^{\mathbf{b}}_{-1}\mathcal{R}^{\mathbf{a}}_{1}(x) = 2(\Delta x)^2$, the ratio of the variance $(\Delta x)^2$ in $|\mathbf{a}, \mathbf{b}; z\rangle_1^{-1}$ to the variance $1/2$ in the vacuum state, shows that there is substantial squeezing (see figure 4).

7. Concluding remarks

In this paper, we have defined a (q, ν)-deformation of the generalized hypergeometric coherent states $|\mathbf{a}, \mathbf{b}; z\rangle_\phi^\nu$ depending on the vector parameters $\mathbf{a}$, $\mathbf{b}$, the function ϕ and the parameter ν. These states are defined such that their normalization function yields generalized basic hypergeometric functions, what justifies the name of generalized basic hypergeometric states (GBHSs). We have found that if $\nu < 0$, the GBHSs are defined on the whole plane, while if $\nu = 0$, they are defined on the unit disc. Limit and particular cases yield well-known non-deformed and deformed states, respectively. Furthermore, we also discussed the associated resolution of unity using Mellin transform techniques. The GBHSs on the whole plane (unit disc) are eigenstates, with eigenvalue z, of suitably defined lowering operators; the latter may be considered as a deformation of the usual annihilation operator of the harmonic oscillator.

In the case where the GBHSs allow the resolution of unity in the form of an ordinary integral with a positive function, the generalized basic hypergeometric coherent states have been used to define a new generalized basic hypergeometric analytic representation of a state $|\varphi\rangle$. We have also studied their self-reproducing property. These deformed coherent states can also be used to define generalized basic hypergeometric Husimi distribution as well as the corresponding phase distribution.

Finally, the physical characteristics of the GBHSs have been analyzed for relevant values of $\mathbf{a}$, $\mathbf{b}$, ϕ and ν. The graphs have showed that the Mandel parameter and the variance ratio ${}^{\mathbf{b}}_{-1}\mathcal{R}^{\mathbf{a}}_{1}(x)$ decrease faster in the case $a_1 = 0, a_2 = q^2, a_i = 0$ for $i \geqslant 3$; $b_1 = q, b_i = 0$ for $i \geqslant 2$; $\phi(q) = 1$ and $\nu = -1$ than the Quesne's case ($\mathbf{a} = -, \mathbf{b} = -$; $\phi(q) = 1$ and $\nu = -1$). In figure 3(*B*), we find that there exists $x_0 \neq 0$ such that the metric factor ${}^{\mathbf{b}}_{-1}\mathcal{W}^{\mathbf{a}}_{1}(x)$ is greater than 1 for $x < x_0$, and ${}^{\mathbf{b}}_{-1}\mathcal{W}^{\mathbf{a}}_{1}(x)$ is less than 1 for $x \geqslant x_0$. Such a physical behavior, which has not been observed in previous works by Quesne, suggests that the (q, ν)-basic hypergeometric states deserve further investigations from both theoretical and experimental aspects in quantum optics.

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